

Number-Theoretic Properties of Generalized Fibonacci and Lucas Sequences
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Abstract

Recurrence sequences occupy an important position in number theory because they provide a natural framework for describing integers generated through simple recursive rules. Among the most well-known examples are the Fibonacci and Lucas sequences, whose terms exhibit rich patterns involving divisibility, greatest common divisors, congruences, identities, and Diophantine equations. The present paper investigates the number-theoretic properties of generalized Fibonacci and Lucas sequences obtained from a second-order linear recurrence relation. A unified formulation is considered so that the classical Fibonacci and Lucas sequences appear as particular cases of the generalized framework. The study focuses on recurrence relations, characteristic roots, closed-form representations, divisibility properties, greatest common divisor relations, addition formulas, Cassini-type identities, and periodic behaviour modulo a positive integer. Particular attention is given to the interaction between the parameters of the recurrence relation and the arithmetic structure of the resulting sequence. Several representative examples are included to illustrate the theoretical results and to demonstrate how the generalized framework extends familiar properties of Fibonacci and Lucas numbers. The results provide a systematic basis for studying recurrence sequences from a number-theoretic perspective and may also be useful in computational problems involving integer sequences, modular arithmetic, and Diophantine equations.

Keywords Generalized Fibonacci sequence, Generalized Lucas sequence, recurrence relations, number theory, divisibility, congruences, greatest common divisor, Cassini identity, Diophantine equations, periodicity.

1. Introduction

Recurrence sequences constitute an important class of discrete mathematical structures and have attracted considerable attention because of their connections with number theory, combinatorics, algebra, and computational mathematics. A recurrence relation generates successive terms from a prescribed set of initial values, and even a simple recurrence may produce intricate arithmetic behaviour. Properties such as divisibility, common divisors,

congruences, periodicity, prime terms, and Diophantine representations provide important avenues for investigating the underlying structure of integer sequences.

Among the classical recurrence sequences, the Fibonacci and Lucas sequences occupy a particularly prominent position. The Fibonacci sequence is defined by

$$F_0 = 0, F_1 = 1, \text{ and } F_{n+2} = F_{n+1} + F_n, n \geq 0.$$

The associated Lucas sequence is given by

$$L_0 = 2, L_1 = 1, \text{ together with the recurrence relation } L_{n+2} = L_{n+1} + L_n.$$

Although these sequences are generated by elementary recurrence relations, their terms exhibit a remarkable variety of arithmetic relationships. Their divisibility properties, greatest common divisor identities, congruence behaviour, recurrence identities, and connections with Diophantine equations have made them important objects of study in number theory. The richness of these properties has also encouraged researchers to investigate broader families of recurrence sequences in which the coefficients and initial conditions are generalized.

A natural framework for such a generalization is provided by the second-order recurrence

$U_{n+2} = P U_{n+1} - Q U_n$, where P and Q are fixed integers. The generalized Fibonacci sequence associated with P and Q is defined by

$$U_0 = 0, U_1 = 1, \text{ whereas the corresponding generalized Lucas sequence is defined by}$$

$V_0 = 2, V_1 = P$, and both sequences satisfy the same recurrence relation. This formulation contains the classical Fibonacci and Lucas sequences as particular cases. Specifically,

$$\text{when } P = 1 \text{ and } Q = -1, \text{ the recurrence becomes } U_{n+2} = U_{n+1} + U_n,$$

and the resulting sequences reduce to the standard Fibonacci and Lucas sequences.

One of the principal advantages of this generalized framework is that it permits several related arithmetic identities to be investigated within a single mathematical setting. The characteristic equation corresponding to the recurrence is

$$x^2 - P x + Q = 0.$$

Let α and β denote its roots. Then

$$\alpha + \beta = P \text{ and } \alpha\beta = Q.$$

When the roots are distinct, the generalized sequences can be represented through Binet-type formulas as

$$U_n = (\alpha^n - \beta^n) / (\alpha - \beta) \text{ and } V_n = \alpha^n + \beta^n.$$

These representations establish a useful relationship between the recurrence sequence and the algebraic properties of its characteristic roots. They are particularly effective in deriving identities and studying the asymptotic behaviour of the sequence. On the other hand, direct recurrence arguments provide a natural approach for establishing integral, divisibility, and congruence properties.

The arithmetic structure of generalized Fibonacci and Lucas sequences is strongly influenced by the parameters P and Q . A change in these parameters may alter the factorization pattern of the terms, their common divisors, their behaviour modulo an integer, and the periodicity of the resulting sequence. Consequently, properties observed for the classical Fibonacci sequence cannot always be transferred directly to the generalized setting. Determining which properties remain valid and identifying the conditions under which they hold therefore constitute meaningful problems in the study of recurrence sequences.

In particular, divisibility and greatest common divisor properties provide a useful means of understanding the relationship between the indices and the corresponding sequence terms. Similarly, generalized versions of well-known identities, such as Cassini-type relations, provide structural information about consecutive terms. The study of these identities can also lead to further questions concerning congruences, primitive divisors, perfect powers, and Diophantine equations involving generalized sequence terms.

Despite the extensive study of classical Fibonacci and Lucas sequences, the generalized framework continues to provide scope for investigating parameter-dependent arithmetic properties. A unified treatment is especially useful because it makes it possible to distinguish results that arise from the general recurrence structure from those that depend on the particular choice $P = 1$ and $Q = -1$. This distinction forms an important motivation for the present study. The main purpose of this paper is therefore to investigate selected number-theoretic properties of generalized Fibonacci and Lucas sequences generated by the recurrence $U_{n+2} = P U_{n+1} - Q U_n$. The study focuses on their fundamental recurrence identities, Binet-type representations, divisibility behaviour, greatest common divisor properties, Cassini-type identities, and congruence and periodic characteristics. The paper also considers representative Diophantine problems involving terms of the generalized sequences. Through these investigations, the paper aims to provide a unified mathematical perspective on the arithmetic structure of generalized Fibonacci and Lucas sequences and to identify directions for further research.

The remainder of the paper is organized as follows. Section 2 introduces the generalized Fibonacci and Lucas sequences and presents the basic definitions and notation required for subsequent analysis. Section 3 develops the characteristic-root formulation and corresponding Binet-type representations. Section 4 establishes fundamental recurrence identities and addition formulas. Section 5 examines divisibility and greatest common divisor properties. Section 6 investigates generalized Cassini-type identities and related algebraic relations. Section 7 discusses congruence and periodic behaviour modulo positive integers. Section 8 considers selected Diophantine equations involving generalized sequence terms. Section 9 provides

numerical illustrations of the principal results. Finally, Section 10 presents the conclusions of the study and outlines several possible directions for future investigation.

2. Preliminaries

Let P and Q be fixed integers. We consider the second-order linear recurrence relation $U_{n+2} = P U_{n+1} - Q U_n$, $n \geq 0$, with the initial conditions $U_0 = 0$, $U_1 = 1$.

The sequence $\{U_n\}_{n \geq 0}$ generated by these conditions is referred to as the **generalized Fibonacci sequence** corresponding to the parameters P and Q . For the same values of P and Q , the associated **generalized Lucas sequence** $\{V_n\}_{n \geq 0}$ is defined by

$$V_0 = 2, V_1 = P, \text{ and } V_{n+2} = P V_{n+1} - Q V_n, n \geq 0.$$

Thus, both sequences satisfy the same second-order recurrence relation, while their initial conditions are different. This common structure provides a convenient framework for establishing relationships between the two sequences and for investigating their arithmetic properties in a unified manner.

The classical Fibonacci and Lucas sequences are obtained as special cases of this generalized formulation. In particular, when

$P = 1$, $Q = -1$, the recurrence relation reduces to $U_{n+2} = U_{n+1} + U_n$. With $U_0 = 0$ and $U_1 = 1$, the resulting sequence is the classical Fibonacci sequence, so that $U_n = F_n$. Similarly, the initial conditions $V_0 = 2$ and $V_1 = 1$ give the classical Lucas sequence, and hence $V_n = L_n$. Therefore, the generalized formulation provides a broader setting in which the familiar properties of Fibonacci and Lucas numbers can be studied together with their extensions.

The characteristic polynomial associated with the recurrence relation is

$$x^2 - P x + Q = 0.$$

Let α and β be its roots. By the relationships between the roots and coefficients of the polynomial, we have $\alpha + \beta = P$ and $\alpha\beta = Q$.

The corresponding discriminant is $\Delta = P^2 - 4Q$. When $\Delta \neq 0$, the characteristic roots are distinct and are given by $\alpha = (P + \sqrt{\Delta})/2$ and $\beta = (P - \sqrt{\Delta})/2$.

In this case, the generalized Fibonacci and Lucas sequences admit the Binet-type representations $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $V_n = \alpha^n + \beta^n$.

These representations establish an important connection between the recurrence sequences and the algebraic properties of their characteristic roots. They are useful for deriving identities, studying growth behaviour, and investigating divisibility and congruence properties.

An important relationship between the generalized Fibonacci and Lucas sequences is

$$V_n = U_{n+1} - Q U_{n-1}, n \geq 1.$$

Indeed, using the Binet-type representations and the relation $\alpha\beta = Q$, we obtain

$$U_{n+1} - Q U_{n-1} = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta) - \alpha\beta(\alpha^{n-1} - \beta^{n-1})/(\alpha - \beta) = \alpha^n + \beta^n = V_n.$$

Hence, the two generalized sequences are not independent; rather, they are closely connected through their common recurrence structure and characteristic roots.

The above formulation establishes the basic mathematical framework required for the subsequent analysis. In particular, it allows the classical Fibonacci and Lucas sequences to be treated as special cases while providing sufficient generality for studying divisibility, greatest common divisors, recurrence identities, congruences, periodicity, and other number-theoretic properties of generalized recurrence sequences.

3. Characteristic Equation and Binet-Type Representation

The characteristic equation plays a fundamental role in the analysis of second-order linear recurrence sequences because it provides an algebraic representation of the recurrence and determines the form of its general solution. For the generalized Fibonacci and Lucas sequences considered in this paper, the recurrence relation is given by

$$U_{n+2} = P U_{n+1} - Q U_n.$$

To obtain the characteristic equation, assume a solution of the form $U_n = r^n$. Substituting this expression into the recurrence relation gives

$$r^{n+2} = P r^{n+1} - Q r^n, \text{ and, after dividing by } r^n, \text{ we obtain} \\ r^2 - P r + Q = 0.$$

Thus, the characteristic equation associated with the generalized recurrence is

$$x^2 - P x + Q = 0.$$

Let α and β denote the two roots of this equation. By Vieta's relations,

$$\alpha + \beta = P \text{ and } \alpha\beta = Q.$$

The discriminant of the characteristic equation is

$$\Delta = P^2 - 4Q.$$

When $\Delta \neq 0$, the roots are distinct and are given by

$$\alpha = (P + \sqrt{\Delta})/2 \text{ and } \beta = (P - \sqrt{\Delta})/2.$$

Since α and β are distinct roots of the characteristic equation, the general solution of the recurrence relation can be expressed as

$$U_n = A\alpha^n + B\beta^n,$$

where A and B are constants determined by the initial conditions. For the generalized Fibonacci sequence, $U_0 = 0$ and $U_1 = 1$. Applying these conditions gives

$$A + B = 0 \text{ and } A\alpha + B\beta = 1.$$

Hence, $A = 1/(\alpha - \beta)$ and $B = -1/(\alpha - \beta)$.

Therefore, the generalized Fibonacci sequence admits the following Binet-type representation:

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta).$$

Similarly, using the initial conditions $V_0 = 2$ and $V_1 = P$ for the generalized Lucas sequence, we obtain $V_n = \alpha^n + \beta^n$.

Thus, the two sequences can be represented in the unified form

$$U_n = (\alpha^n - \beta^n)/(\alpha - \beta) \text{ and } V_n = \alpha^n + \beta^n.$$

These formulas provide an algebraic description of the generalized Fibonacci and Lucas sequences and are particularly useful for deriving identities involving their terms. For example, since $\alpha\beta = Q$, the Binet-type representations can be used to establish relationships between U_n and V_n , as well as to derive addition formulas and Cassini-type identities.

The nature of the characteristic roots depends on the value of the discriminant Δ . If $\Delta > 0$, the roots α and β are distinct real numbers. If $\Delta = 0$, the characteristic equation has a repeated root, namely $\alpha = \beta = P/2$. In this case, the standard Binet-type expression containing $\alpha - \beta$ in the denominator is not directly applicable, and the recurrence must be treated using the repeated-root form. When $\Delta < 0$, the characteristic roots are complex conjugates, provided P and Q are real. Nevertheless, because the recurrence has integer coefficients and integer initial values, the resulting sequence terms remain integers. For the distinct-root case, the Binet-type formulas also reveal the connection between the growth of the sequence and the magnitude of the characteristic roots. In particular, when one root has a greater absolute value than the other, the term associated with the dominant root determines the asymptotic behaviour of the sequence. This observation is useful when studying the growth of sequence terms and equations involving large indices.

For the classical Fibonacci and Lucas sequences, $P = 1$ and $Q = -1$. Consequently, the characteristic equation becomes

$$x^2 - x - 1 = 0, \text{ whose roots are } \alpha = (1 + \sqrt{5})/2 \text{ and } \beta = (1 - \sqrt{5})/2.$$

Therefore, the generalized formulas reduce to the familiar expressions

$$F_n = (\alpha^n - \beta^n)/(\alpha - \beta) \text{ and } L_n = \alpha^n + \beta^n.$$

Hence, the characteristic-root approach not only provides a convenient method for analysing generalized recurrence sequences but also demonstrates how the classical Fibonacci and Lucas formulas arise naturally as particular cases of the general framework.

4. Fundamental Identities

The generalized Fibonacci and Lucas sequences possess a rich collection of algebraic identities arising from their common recurrence relation and characteristic roots. These identities provide useful tools for deriving divisibility results, establishing relationships between terms with different indices, and investigating further arithmetic properties. In this section, several fundamental identities are developed within the generalized framework.

From the Binet-type representations $U_n = (\alpha^n - \beta^n)/(\alpha - \beta)$ and $V_n = \alpha^n + \beta^n$,

together with the relation $\alpha\beta = Q$, we obtain an important connection between the two sequences. For $n \geq 1$, $V_n = U_{n+1} - Q U_{n-1}$.

$$\begin{aligned} \text{Indeed, } U_{n+1} - Q U_{n-1} &= (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta) - \alpha\beta (\alpha^{n-1} - \beta^{n-1})/(\alpha - \beta) \\ &= [\alpha^n (\alpha - \beta) + \beta^n (\alpha - \beta)]/(\alpha - \beta) \\ &= \alpha^n + \beta^n \\ &= V_n. \end{aligned}$$

Thus,

$$V_n = U_{n+1} - Q U_{n-1}. \quad (4.1)$$

Using the recurrence relation $U_{n+1} = P U_n - Q U_{n-1}$, equation (4.1) can also be expressed as

$$V_n = P U_n - 2Q U_{n-1}, \quad n \geq 1. \quad (4.2)$$

These relations demonstrate that the generalized Lucas sequence can be completely connected with the generalized Fibonacci sequence through the same recurrence parameters P and Q .

A further important identity is obtained by considering two non-negative integers m and n .

For, $m \geq 1$ and $n \geq 0$, the generalized Fibonacci sequence satisfies the addition formula

$$U_{m+n} = U_m U_{n+1} - Q U_{m-1} U_n. \quad (4.3)$$

This formula can be established directly from the Binet-type representation. Substituting

$$\begin{aligned} U_m &= (\alpha^m - \beta^m)/(\alpha - \beta) \text{ and } U_{n+1} = (\alpha^{n+1} - \beta^{n+1})/(\alpha - \beta), \text{ and using } \alpha\beta = Q, \text{ we obtain} \\ U_m U_{n+1} - Q U_{m-1} U_n &= [(\alpha^m - \beta^m)(\alpha^{n+1} - \beta^{n+1}) - \alpha\beta (\alpha^{m-1} - \beta^{m-1})(\alpha^n - \beta^n)]/(\alpha - \beta)^2. \end{aligned}$$

After simplification,

$$U_m U_{n+1} - Q U_{m-1} U_n = (\alpha^{m+n} - \beta^{m+n})/(\alpha - \beta) = U_{m+n}. \text{ Hence, equation (4.3) follows.}$$

Another useful form of the addition formula is

$$U_{m+n} = U_m U_{n+1} - Q U_{m-1} U_n = U_n U_{m+1} - Q U_{n-1} U_m. \quad (4.4)$$

The equality of these two expressions reflects the symmetry between the indices m and n and is useful in the analysis of divisibility and greatest common divisor properties.

The generalized Lucas sequence also satisfies an important addition relation. For suitable integers m and n with $m \geq n$,

$$\begin{aligned} V_{m+n} &= V_m V_n - Q^n V_{m-n}. \quad (4.5) \text{ This identity follows from } V_m \\ V_n &= (\alpha^m + \beta^m)(\alpha^n + \beta^n) \\ &= \alpha^{m+n} + \beta^{m+n} + \alpha^m \beta^n + \alpha^n \beta^m. \end{aligned}$$

Since $\alpha\beta = Q$, $\alpha^m \beta^n + \alpha^n \beta^m = Q^n (\alpha^{m-n} + \beta^{m-n}) = Q^n V_{m-n}$.

Therefore, $V_m V_n = V_{m+n} + Q^n V_{m-n}$,

which gives $V_{m+n} = V_m V_n - Q^n V_{m-n}$.

A particularly useful relation connecting the two sequences is

$$V_n^2 - (P^2 - 4Q) U_n^2 = 4Q^n. \quad (4.6)$$

To verify this identity, observe that

$$V_n^2 - (P^2 - 4Q) U_n^2 = (\alpha^n + \beta^n)^2 - (\alpha - \beta)^2$$

$$(\alpha^n - \beta^n)^2 / (\alpha - \beta)^2 = (\alpha^n + \beta^n)^2 - (\alpha^n - \beta^n)^2 = 4\alpha^n\beta^n = 4(\alpha\beta)^n = 4Q^n.$$

Hence, equation (4.6) is established.

The above identities form an important algebraic foundation for the subsequent study of generalized Fibonacci and Lucas sequences. In particular, the addition formulas can be used to investigate divisibility and greatest common divisor properties, while equation (4.6) provides a useful connection with quadratic Diophantine equations. When $P = 1$ and $Q = -1$, these generalized identities reduce to several well-known identities involving the classical Fibonacci and Lucas numbers. Thus, the generalized framework preserves the essential structure of the classical sequences while allowing a broader range of number-theoretic properties to be examined.

5. Divisibility Properties

Divisibility provides one of the most important ways of understanding the arithmetic structure of recurrence sequences. In the case of the classical Fibonacci sequence, the relationship between the divisibility of indices and the divisibility of corresponding terms is particularly well known. The generalized Fibonacci sequence offers a broader setting in which similar properties can be investigated while taking into account the influence of the recurrence parameters P and Q .

Consider the generalized Fibonacci sequence defined by

$$U_{n+2} = P U_{n+1} - Q U_n, \text{ with } U_0 = 0, U_1 = 1.$$

The addition identity established in the previous section,

$$U_{m+n} = U_m U_{n+1} - Q U_{m-1} U_n,$$

provides a useful starting point for analysing divisibility relations among its terms. In particular, it connects a term with index $m + n$ to terms whose indices are m and n . This structure can be exploited repeatedly to study sequence terms whose indices have a prescribed divisibility relationship.

Under appropriate conditions on the parameters P and Q , the generalized Fibonacci sequence has the structure of a divisibility sequence. In such cases, if

$$m \mid n, \text{ then } U_m \mid U_n. \quad (5.1)$$

To see the significance of this property, suppose that $n = km$ for some positive integer k . The addition formula can then be applied successively to express U_n in terms of U_m and other integer sequence terms. Under the required parameter conditions, this leads to the divisibility relation in (5.1). Thus, divisibility among the indices is reflected directly in divisibility among the corresponding sequence terms.

The divisibility property becomes particularly useful when combined with the greatest common divisor of the indices. For suitable generalized Fibonacci sequences, one may obtain the stronger relationship

$$\gcd(U_m, U_n) = U_{\gcd(m,n)}. \quad (5.2)$$

This identity establishes a close correspondence between the arithmetic structure of the indices and that of the sequence terms. It also provides an efficient method for determining common factors of two generalized Fibonacci numbers without computing their complete factorizations.

The divisibility behaviour of the generalized sequence depends significantly on the parameters P and Q . Consequently, the relation

$$m \mid n \Rightarrow U_m \mid U_n$$

should not be assumed for every arbitrary second-order recurrence. Additional restrictions may be required to ensure that the sequence possesses the desired divisibility structure. This parameter dependence is one of the important aspects that distinguishes the generalized setting from the classical Fibonacci case.

Another useful consequence arises when the index itself has a factorization. If

$$n = rs, \text{ then, whenever the divisibility property is satisfied, } U_r \mid U_n.$$

Thus, the factorization of the index provides immediate information about possible factors of the corresponding sequence term. This observation can be useful in studying the factorization pattern of generalized Fibonacci numbers and in identifying terms that necessarily possess non-trivial divisors.

The addition formulas also provide a mechanism for investigating divisibility when the index difference is considered. For example, if $m > n$, then

$$U_m = U_n U_{m-n+1} - Q U_{n-1} U_{m-n}. \quad (5.3)$$

This representation shows how a term with a larger index can be expressed through terms associated with smaller indices. Such relations are useful in recursive arguments and can also contribute to proofs concerning common divisors.

The divisibility structure is closely related to the occurrence of prime terms. If n is composite, say

$$n = rs, \text{ with } r, s > 1, \text{ and if the sequence satisfies the divisibility property, then } U_r \mid U_n.$$

Therefore, except in special situations, a term U_n corresponding to a composite index is unlikely to behave like a completely new prime term. This observation motivates the study of prime indices and primitive divisors when investigating prime values within generalized recurrence sequences.

Similarly, perfect powers among generalized Fibonacci and Lucas terms form an interesting class of Diophantine problems. One may consider equations such as

$$U_n = y^k, \text{ where } y \text{ is an integer and } k \geq 2.$$

The exponential growth of recurrence sequences, together with their divisibility and congruence properties, can impose strong restrictions on possible solutions. Consequently, the study of perfect powers provides a natural connection between recurrence sequences and Diophantine number theory.

The parameters P and Q also influence the factorization and divisibility behaviour of individual terms. Different choices of these parameters can produce sequences with significantly different arithmetic patterns. Hence, an important part of the generalized theory is to determine parameter conditions under which properties such as divisibility, greatest common divisors, primitive divisors, and factorization patterns remain valid.

In summary, the divisibility properties of generalized Fibonacci sequences reveal a strong interaction between the additive structure of the indices and the multiplicative structure of the sequence terms. The principal questions arising from this analysis include determining the conditions under which

$$m \mid n \Rightarrow U_m \mid U_n,$$

establishing corresponding greatest common divisor formulas, identifying the role of P and Q , and investigating the occurrence of prime and perfect-power terms. These aspects provide a natural foundation for the subsequent study of the arithmetic properties of generalized Fibonacci and Lucas sequences.

6. Cassini-Type Identities

Cassini's identity is one of the most recognizable properties of the Fibonacci sequence:

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

For the generalized sequence, a corresponding identity can be derived from the recurrence relation. Using the characteristic-root representation, one obtains

$$U_{n+1}U_{n-1} - U_n^2 = -Q^n.$$

Equivalently,

$$U_{n+1}U_{n-1} - U_n^2 = -Q^n.$$

When $P = 1$ and $Q = -1$, this reduces to

$$F_{n+1}F_{n-1} - F_n^2 = (-1)^n.$$

Thus, the classical Cassini identity appears naturally as a special case of the generalized result. A related identity involving the generalized Lucas sequence is

$$V_n^2 - (P^2 - 4Q) U_n^2 = 4Q^n.$$

This relation connects the generalized Fibonacci and Lucas sequences and provides an important algebraic tool for investigating Diophantine equations.

7. Congruence Properties and Periodicity

Modular arithmetic provides another useful approach to studying recurrence sequences. Let m be a positive integer. Since the recurrence relation determines each term from a fixed number of preceding terms, the sequence of ordered pairs

$(U_n, U_{n+1}) \bmod m$ can take only finitely many values.

Consequently, the sequence eventually becomes periodic modulo m . Under suitable conditions, the sequence may be purely periodic from the beginning. For the classical Fibonacci sequence, this phenomenon is associated with Pisano periods. In the generalized setting, the corresponding period depends on the parameters P, Q and the modulus m .

8. Diophantine Equations Involving Generalized Sequences

Generalized Fibonacci and Lucas sequences also provide a natural source of Diophantine equations. A typical problem is to determine integer solutions of equations involving one or more sequence terms, such as

$$U_n = k, \text{ or } U_n = x^2, \text{ or } U_n = V_m.$$

More generally, one may consider equations of the form

$$U_n + U_m = U_r \text{ and } U_n U_m = U_r,$$

where the indices and parameters are subject to appropriate restrictions.

The exponential growth of recurrence sequences often imposes strong restrictions on the possible solutions of such equations. Binet-type representations can be used to estimate the size of terms, while congruence arguments can eliminate impossible cases.

A systematic study of these equations can therefore combine recurrence theory with elementary and algebraic number theory.

9. Numerical Illustration

Consider the parameters $P = 3, Q = 1$.

The generalized Fibonacci sequence is given by

$$U_0 = 0, U_1 = 1, \text{ and } U_{n+2} = 3U_{n+1} - U_n.$$

The first few terms are

$$U_0 = 0, U_1 = 1, U_2 = 3, U_3 = 8, U_4 = 21, U_5 = 55, U_6 = 144.$$

The corresponding generalized Lucas sequence begins with

$$V_0 = 2, V_1 = 3, \text{ and } V_{n+2} = 3V_{n+1} - V_n.$$

Therefore, $V_2 = 7, V_3 = 18, V_4 = 47, V_5 = 123$.

For $n = 3$, the generalized Cassini-type identity gives

$$U_4 U_2 - U_3^2 = 21(3) - 8^2$$

$$= 63 - 64 = -1.$$

Since $Q = 1$, $-Q^3 = -1$, and therefore, the identity is verified.

This example illustrates how the generalized recurrence framework produces sequences with arithmetic patterns analogous to, but distinct from, the classical Fibonacci sequence.

10. Discussion

The preceding analysis shows that generalized Fibonacci and Lucas sequences form a flexible class of integer sequences with a rich arithmetic structure. The parameters P and Q play a central role in determining the behaviour of the sequence. They influence the characteristic roots, growth rate, divisibility pattern, congruence properties, and identities satisfied by the sequence terms. One important feature of the generalized formulation is that several classical results can be obtained as special cases. This provides a unified perspective and avoids treating related recurrence sequences as completely separate objects.

At the same time, generalization introduces new questions. A property that holds for the classical Fibonacci sequence may require additional restrictions in the generalized setting. Consequently, identifying the exact parameter conditions under which a particular property remains valid is an important mathematical problem.

From a computational perspective, generalized recurrence sequences are also convenient for experimentation. Terms can be generated efficiently, allowing conjectures concerning divisibility, periodicity, and congruence patterns to be tested for a wide range of parameter values. Such computational exploration can serve as a useful preliminary step before establishing results analytically.

11. Possible Research Problems

The generalized framework considered in this paper suggests several problems for further investigation:

1. Determine sufficient conditions on P and Q under which the generalized Fibonacci sequence forms a divisibility sequence.
2. Investigate the exact greatest common divisor relationship between U_m and U_n for different classes of parameters.
3. Study the distribution of prime and composite terms in generalized Fibonacci and Lucas sequences.
4. Determine the modular periods of U_n and V_n for selected classes of P , Q and modulus m .
5. Investigate Diophantine equations involving generalized Fibonacci and Lucas numbers.
6. Study primitive divisors of generalized recurrence terms.
7. Investigate perfect powers occurring among generalized sequence terms.

8. Explore generalized identities analogous to Cassini's, Catalan's, and d'Ocagne's identities.
9. Develop computational algorithms for detecting divisibility and periodicity patterns.
10. Examine connections between generalized recurrence sequences and Pell-type equations.

12. Conclusion

This paper has presented a unified study of the number-theoretic properties of generalized Fibonacci and Lucas sequences. By considering the second-order recurrence

$U_{n+2} = P U_{n+1} - Q U_n$, the classical Fibonacci and Lucas sequences can be incorporated into a much broader family of recurrence sequences. The characteristic equation and its roots provide Binet-type representations, while the recurrence itself offers an effective mechanism for deriving arithmetic identities.

The analysis has highlighted several important aspects, including fundamental recurrence identities, divisibility behaviour, greatest common divisor relationships, Cassini-type formulas, congruence properties, periodicity, and Diophantine problems. These properties demonstrate that generalized recurrence sequences possess a substantial number-theoretic structure and provide a useful framework for studying both classical and newly defined integer sequences. An important direction for future research is to identify precise conditions on the recurrence parameters under which stronger divisibility, gcd, periodicity, and Diophantine properties hold. Computational experiments can also be combined with theoretical arguments to formulate and verify new conjectures. Thus, generalized Fibonacci and Lucas sequences continue to offer a productive area for research at the intersection of recurrence theory, number theory, and computational mathematics.

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