NJESR/June-2026/Volume-8/Issue-6.. E-ISSN 2582-5836
DOI-10.53571/NJESR.2026.8.6. 6 -9

Discussion for solving singular boundary value Problem by taking the fourth order finite diffence method directly

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(Received-15 May2026/Revised-28May2026/Accepted-1June2026/Published-6June 2026)

Abstract: The finite difference methods are always a Convenient-choice for solving boundary value problems, because of their simplicity. The fourth order finite difference method is then employed to solve the boundary value problem. By stabilizing the classical central difference (CD) method. we devlop a fourth order finite difference method.

To obtain this method, we reapproximate the CD approximation by rewriting its error term as a combination of first and second derivative terms.

Keywords:- Boundary value problem, finite difference method 2nd ordere difference method, forth order difference method approximation, error.

Introduction:- Differential equations plays major role in applications of sciences and engineering. It arises in wide variety of engineering applications for example electromagnetic theory, signal processing, computational fluid dynamics. There is a considerable interest on numerical methods on Singular boundary value problems,

The study on the linear difference equations with variables general much in common. In this chapter, we discuss a direct method for solving singular boundary value problem. The finite difference methods are always a convenient choice for solving boundary value problems, because of their simplicity. The original differential equation is modified at the singular point. The fourth order finite difference method is then employed to solve the boundary value problem. By stabilizing the classical central difference (CD) method, we develop a fourth order finite difference method. To obtain this method, we re-approximate the CD approximation by rewriting its error terms as a combination of first and second derivative terms and approximating them.[16].

2.2 Formulations:

We consider a singular two-point boundary value problem given by

$$Ly \equiv y''(x) + \frac{\kappa}{x} y'(x) + q(x)y(x) = r(x), \quad (2.1)$$

$$y'(0) = 0, \quad (2.2)$$

$$\mathbf{Y(1) =} \quad (2.3)$$

Jamet has shown that for Eq. (2.1) the derivative boundary condition is imposed due to nature of physical situation of the problem. Due to the singularity at $x = 0$, we modify the problem near the singular point. To set up difference equation of (2.1) divide [0,1] into n equal parts, each of the length h, we have $x_i = ih, i = 0,1,\ldots,n$. For simplicity, let $q(x_i)=q_i$, $r(x_i)=r_i$, $y(x_i)=y_i$, $y'(x_i)$ y'_i and $y''(x_i)=y''_i$.

Since x = 0 is singular point of Eq. (2.1), we first modify Eq. (2.1) at x = x0 =0 as follows:

$$y''(0) + \underset{x \to 0}{Lt} \frac{k}{x} h'(x) + q(0)y(0) = r(0)$$

Using L. Hospital rule, we have

$$\underset{x \to 0}{Lt} \frac{k}{x} y'(x) = ky''(0)$$

Than we obtain

$$(1+k)y''(x) + q(x)y(x) = r(x) \text{ at } x = 0$$

Now? we describe a fourth order finite difference method, which leads to a tridiagonal system, which can be solved by Thomas Algorithm. By Taylor series expansion we obtain the CD formulas for y', y'' assuming that y has continuous fourth order derivaties in the interval [0,1]:

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)}(\xi) \tag{2.4}$$

$$y_i' \cong \frac{y_{i+1} - y_{i-1} + y_{i-1}}{2h} - \frac{h^2}{6} y^{(m)}(\eta) \tag{2.5}$$

where $\xi, \eta \in [x_{i-1}, x_{i-1}]$. Substituting (2.4) and (2.5) in (2.1) at $x = x_i$, we get the CD operator Lh, defined by $_{1,} x$

$$L_h y_i \equiv a_i y_{i+1} - b_i y_i + c_i y_{i-1} = d_i + \tau_i[y] 1 \le i \le n-1, \tag{2.6}$$

Where

$$a_i = \frac{1}{h^2} + \frac{k}{2hx_1}, \; b_i = \frac{2}{h^2} - q_i, \; c_i = \frac{1}{h^2} - \frac{k}{2hx_1}, \; d_i = r_i \tag{2.7}$$

And

$$\tau_i[y] = \frac{h^2}{12} h^{(4)}(\xi) + \frac{h^2 k}{6x_i} y^{m}(\eta),$$

Where ξ, η $[x_{i-1}\ x_{i+1}]$ .x here τ [y]are local truncation errors of the CD approximation. To obtain numeric solution of (2.7-2.8) by the CD operator Lh, we solve the system of equations formed by the three-term recurrence relation:

$$L_h y_i \equiv a_i y_{i+1} - b_i y_i + c_i y_{i-1} = d_i, 1 \le i \le n-1. \tag{2.8}$$

By rewriting the CD formulas for y', y'' in new form as given below: $y \quad 2y \quad y$

$$y_i'' \cong \frac{y_{i+1} - 2y_i + y_{i-1}}{h^2} - \frac{h^2}{12} y^{(4)} + R_1, \tag{2.9}$$

$$y_i' \cong \frac{y_{i+1} - y_{i-1}}{2h} - \frac{h^2}{6} y''' + R_2, \tag{2.10}$$

Where

$$R_1 = -\frac{2h^4 y^{(6)}(\xi)}{6!} \text{ and } R_2 = \frac{h^4 y^{(5)}(\eta)}{5!}$$

for $\xi, \eta \in [x_{i-1}, x_{i-1}]$. Substituting these $y_i^{'}, y_i^{''}$ from Eqs.(2.9) and (2.10) in (2.1) at $x = x_i$, we get the CD approximation in a form that includes all the O(h2) error terms:

$$L_h y_i - \frac{h^2}{12}\left(2\frac{k}{x_i} y_i^{'''} + y_i^{(4)}\right) + \bar{R} = r_i, \tag{2.11}$$

Lh is CD operator given as in (2.5) and $R = R_1(k/x_i)R_2$. By writing $q(x) = q, r(x) = r,$ in (2.1) we obtain.

$$y^{''} = r - qy - \frac{k}{x} y^{'}.$$

Differentiating above equation with respect to x, we obtain

$$y^{''} = r^{'} - \left[\frac{k}{x} y^{''} + \left(q - \frac{k}{x^2}\right) y^{'} + q^{'} y\right]. \tag{2.12}$$

Now differentiating (2.12) with respect to x, we get

$$y^{(4)} = r^{''} - \left[\frac{k}{x} y^{m} + \left(q - 2\frac{k}{x^2}\right) y^{'} + \left(2\frac{k}{x^3} + 2q^{'}\right) y^{'} + q^{'''} y\right], \tag{2.13}$$

then

$$2\frac{k}{x} y^{m} + y^{(4)} = \left[2\frac{k}{x^2} - \frac{k^2}{x^2} - q\right] y^{n} + \left[-\frac{k}{x}\left(q - \frac{k}{x^2}\right) - 2\frac{k}{x^3} - 2q'\right] y' \tag{2.14}$$

$$-\left[q^{n} + \frac{k}{x} q'\right] y + \frac{k}{x} r' + r''.$$

Substituting (2.14) in (2.11), we get the equation

$$L_h y_i - \frac{h^2}{12}\left[\left(2\frac{k}{x_i^2} - \frac{k^2}{x_i^2} - q_i\right) y_i^{''} + \left(-\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) - 2\frac{k}{x_i^3} - 2q_i^{'}\right) y_i^{'} \left(q_i^{''} + \frac{k}{x_i} q_i^{'}\right) y_i\right]$$

$$+ R = r_i + \frac{h^2}{12}\left(\frac{k}{x_i} r_i^{'} + r_i^{''}\right). \tag{2.15}$$

we approximate the converted error terms in Eq. (2.15) by using for $_i y^{''}$ and $_i y^{'}$ from Eqs. (2.4) and (2.5). Then adding these new approximations to $L\ y$ defined by (2.6) and (2.7), we obtain the fourth order operator.

Where

$$a_i^* = a_i - \frac{1}{12}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\Big|\right] + \frac{h}{24}\left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q'\Big|\right],$$

$$b_i^* = b_i - \frac{1}{6}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\Big|\right] + \frac{h^2}{12}\left[q_i'' + \frac{k}{x_i}q_i'\Big|\right],$$

$$c_i^* = c_i - \frac{1}{12}\left[2\frac{k}{x_i} - \frac{k^2}{x_i^2} - q_i\Big|\right] - \frac{h^2}{24}\left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q'\Big|\right],$$

$$d_i^* = d_i - \frac{h^2}{12}\left[\frac{k}{x_i}r_i' + r_i''\Big|\right].$$

Here a , b ,ci, di are given in (2.7) and $\tau^*[y]$ are the local truncation errors of the Eq. (2.16), given by

$$\tau_i^*[y] = -\left[2\frac{k}{x_i^2} - \frac{k^2}{x_i^2} - q_i\Big|\right]\frac{h^4}{144}y^{(4)} + \left[\frac{k}{x_i}\left(q_i - \frac{k}{x_i^2}\right) + 2\frac{k}{x_i^3} + 2q_i'\Big|\right]\frac{h^4}{72}y_i^{m} -$$

Where R=R+(k/x)R = O(h^4) We solve the system of equations formed by the three- recurrence relationship:

$$L_h^* y_i \equiv a_i^* y_{i+1} - b_i^* y + c_i^* y_{i-1} = d_i^* \qquad 1 \le i \le n-1. \qquad (2.17)$$

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